p-adic methods in string theory

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Abstract: We briefly describe string theory and p-adic string theory and their purposes. Next, we introduce some necessary mathematical machinery. This is used to explain a regularization of a product formula due to Freund and Witten with unclear meaning. Finally, we prove the main formula and conclude with its usage.

I. INTRODUCTION

Paraphrasing D. Tong (cf. [8]), string theory is an all-encompassing theory of the universe which intends to unify all the forces of nature in a single quantum mechanical framework. Even though presently we are yet to find experimental evidence of string theory, it has given impressive insights to important problems in particle physics and cosmology. It is also one of the foremost examples of physics suggesting new directions in mathematics, for example **mirror symmetry**, related to Calabi-Yau manifolds.

Bearing in mind the motivation to study this theory, we delve into p-adic string theory. Physical models have traditionally worked on the archimedian completion of $\mathbb{Q}$, but we can also consider the p-adic ones [6]. The physical problem of the choice of the prime p is dealt with adeles, making the theory independent of this choice. One can interpret a p-adic string simply as a string whose world-sheet is parametrized by p-adic numbers and start defining the analog p-adic Veneziano amplitude and proceed from there, as in [9].

p-adic string theory provides us with a useful toy-model which allows us to check important conjectures on tachyons because computations can be more easily carried out. Moreover, the lagrangian for this type of particles is very well understood and easy to work with. However, the theory fails, as of yet, to properly explain supersymmetry, though that does not invalidate the previous points; cf. [3].

A product formula was proposed by Freund and Witten in [2] relating the associated p-adic amplitude to strings with the euclidean amplitude for tachyons in a set of Mandelstam variables. However, this formula had no clear meaning since the product diverged for every possible value for the Mandelstam variables. The purpose of this present thesis is to explain how this formula can be regularized and thus we will be able to retrieve information from the usual amplitude via the p-adic ones as with other product formulae in Number Theory.

II. THE p-ADIC NUMBERS

We will start with the introduction of the ring of the p-adic integers, $(\mathbb{Z}_p, +, \cdot)$. A p-adic integer x is a formal sum $x = \sum_{i\geq 0} a_i p^i$, with $0 \leq a_i < p$ for all $i \geq 0$, and $a_i \in \mathbb{Z}$. The $+$ operations is thus defined componentwise taking into account the restrictions for the new coefficients, so that the carryover can have an effect for all i. The $\cdot$ operation is defined similarly with the needed carryovers, making $\mathbb{Z}_p$ into a ring.

We define an absolute value on $\mathbb{Z}_p$ by $|x|_p := p^{-v}$, where v is the first natural subindex for which $a_i \neq 0$ and $|0|_p := 0$. This will induce a metric on $\mathbb{Z}_p$, thus making it into a topological space. Moreover, this absolute value is nonarchimedean.

It is straightforward to check that this topology is compatible with the operations defined, that is, they are continuous with this topology. Consequently, $(\mathbb{Z}_p, +)$ is a topological group and $(\mathbb{Z}_p, +, \cdot)$ is a topological ring.

We can easily check that $\mathbb{Z}_p$ is an integral domain by using $|\cdot|_p$. Since the p-adic numbers can be also constructed as the inverse limit of the projective system $(\mathbb{Z}/p^n\mathbb{Z}, \pi_{n,m})$, $\pi_{n,m}$ being the canonical projection $\pi_{n,m} : \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^m\mathbb{Z}$, $n \geq m$, and each of these are finite rings, therefore compact topological rings, a theorem ensures us that $\mathbb{Z}_p$ is also a compact topological ring.

On the other hand, since we know $\mathbb{Z}_p$ is an integral domain, we can also consider its field of fractions, denoted by $\mathbb{Q}_p$, that one can check is isomorphic to $\mathbb{Z}_p[1/p]$. It is readily checked that the previous p-adic absolute value can be extended naturally to $\mathbb{Q}_p$ and also that an element $x \in \mathbb{Q}_p$ admits a Laurent-type expansion as $\sum_{i\geq i_0} a_i p^i$, $i_0 \in \mathbb{Z}$. We will call the fractional part of $x \in \mathbb{Q}_p$, when i_0 is negative, the sum $\{x\}_p = \sum_{i=i_0}^{-1} a_i p^i$. Furthermore, by Ostrowski's theorem, $\mathbb{Q}_p$ is the only nonarchimedian completion of $\mathbb{Q}$ which extends $|\cdot|_p$. We will make use of the following decompositions:

$$\mathbb{Q}_p = \bigcup_{m\geq 0} p^{-m}\mathbb{Z}_p \quad \text{and} \quad \mathbb{Z}_p = \biguplus_{m\geq 0} p^m\mathbb{Z}_p \setminus p^{m+1}\mathbb{Z}_p.$$

These decompositions together with the fact that $\mathbb{Z}_p$ is compact give us that $(\mathbb{Q}_p, +)$ is a locally compact topological group. Notice that $\mathbb{Z}_p^* = \mathbb{Z}_p \setminus p\mathbb{Z}_p$, which is easy to prove again using $|\cdot|_p$.

Considering D a squarefree integer and $\mathbb{Q}_p(\sqrt{D})$ the extension of $\mathbb{Q}_p$, and looking at the ramification properties of integral primes in $\mathbb{Q}(\sqrt{D})$, we define $q = q_{D,p}$ to be 2 if $p = 2$ and $D \not\equiv 5 \pmod 8$ and 2^2 if $p = 2$ and $D \equiv 5 \pmod 8$. If $p \neq 2$, it is p if D is a square in $\mathbb{Q}_p^*$ or

$|D|_p = \frac{1}{p}$, and p^2 if D is not a square in $\mathbb{Q}_p^*$ and $|D|_p = 1$. Similarly, v is defined to be 2 if D is a square in $\mathbb{Q}_p$ and 1 otherwise. Lastly, for $\alpha \in \mathbb{C}$ we define

$$\rho_{D,p}(\alpha) = \begin{cases} 1, & \text{if } |D|_p = 1, p \neq 2 \\ & \text{or } D \equiv 1 \pmod 4, p = 2, \\ p^{\alpha - \frac{1}{2}}, & \text{if } |D|_p = \frac{1}{p}, p \neq 2, \\ 2^{2\alpha-1}, & \text{if } D \equiv 3 \pmod 4, p = 2, \\ 2^{3\alpha - \frac{3}{2}}, & \text{if } |D|_2 = \frac{1}{2}, p = 2. \end{cases}$$

If we consider the field $\mathbb{Q}$ and its p-adic completions for all p, we say that an **adele** of $\mathbb{Q}$ is an infinite sequence $x = (x_\infty, x_2, x_3, \dots, x_p, \dots)$, with $x_p \in \mathbb{Q}_p$, denoting $\mathbb{Q}_\infty = \mathbb{R}$, and such that there exists a prime $P(x)$ with $x_p \in \mathbb{Z}_p$ for all $p > P(x)$. The set of adeles forms the **adele ring** $\mathbb{A}_\mathbb{Q}$ with the $+$ and $\cdot$ operations introduced component-wise.

$\mathbb{A}_\mathbb{Q}$ is also a topological ring with basis of open neighbourhoods of zero given by subrings of the form

$$A^{(P)} = \{x \in \mathbb{A}_\mathbb{Q} : x_p \in \mathbb{Q}_p \text{ all } p,\ x_p \in \mathbb{Z}_p \quad p > P\},$$

for all naturals P. We also have that $\mathbb{A}_\mathbb{Q}$ is a locally compact topological ring. Adeles for the field $\mathbb{Q}(\sqrt{D})$, where D is a squarefree integer, will also be introduced as follows: we divide the prime numbers in two infinite disjoint sets (this can be shown via the Legendre symbol) P_D^+ if D is a square in $\mathbb{Z}/p\mathbb{Z}$ (including 0) and P_D^- otherwise. An **adele** z of the field $\mathbb{Q}(\sqrt{D})$ is, thus, an infinite sequence of the form $z = (z_\infty, z_2, \dots, z_p, \dots)$, where we put

$$z_\infty = \begin{cases} (x_\infty, x'_\infty), & \text{if } D > 0,\ x_\infty, x'_\infty \in \mathbb{R}, \\ z_\infty, & \text{if } D < 0, z_\infty \in \mathbb{C}, \end{cases}$$

$$z_p = \begin{cases} (x_p, x'_p), & \text{if } p \in P_D^+, x_p, x'_p \in \mathbb{Q}_p \\ z_p, & \text{if } p \in P_D^-, z_p \in \mathbb{Q}_p(\sqrt{D}), \end{cases}$$

and where we also require the existence of a natural $P = P(z)$ such that $x_p, x'_p \in \mathbb{Z}_p$ for $p \in P_D^+, p > P$, and $z_p \in \mathbb{Z}_p(\sqrt{D})$ when $p \in P_D^-, p > P$. The set of adeles also forms a ring $\mathbb{A}_D$ where a topology can also be introduced in a similar way to the adeles of $\mathbb{Q}$.

III. TOOLS AND RESULTS USED

In this section we will introduce tools that will be used for the regularization of product of amplitudes formulae.

A. Gamma functions and Fourier transforms on $\mathbb{R}$ and $\mathbb{C}$

We begin recalling the definition and some basic facts about the complex Euler Gamma function:

$$\Gamma(\alpha) := \int_0^\infty t^{\alpha-1} e^{-t} dt, \text{ Re } \alpha > 0.$$

We note that this function has the functional equation $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, so that considering it with a complex variable, it will be well defined by the principle of analytic continuation on all $\mathbb{C}$ except for the nonpositive integers (including 0). We also have the useful reflection formula by Euler $\Gamma(\alpha)\Gamma(1-\alpha) = \frac{\pi}{\sin \pi\alpha}$.

In a similar spirit, and to describe the Fourier transform of the useful homogeneous distribution $|x|^{\alpha-1}$, we define the Gel'fand-Graev gamma function of $\mathbb{R}$ as

$$\Gamma_\infty(\alpha) := \int_{-\infty}^\infty |x|^{\alpha-1} \exp(-2\pi i x) dx,$$

which can be shown to fulfill $\Gamma_\infty(\alpha)\Gamma_\infty(1-\alpha) = 1$. Now, just as with the usual Beta function, the Gel'fand-Graev beta function, used to express the convolution of $|x|^{\alpha-1}$ and $|x|^{\beta-1}$ for a tubular domain, is $B_\infty(\alpha, \beta) = \Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)$, where $\alpha + \beta + \gamma = 1$, which ties to the Mandelstam variables constraint. Those functions have useful symmetric representations, cf. [1]. For the field $\mathbb{C}$, as a completion of $\mathbb{Q}(\sqrt{D})$ for $D < 0$, we also have a similar definition of Fourier transform and thus, of the Gel'fand-Graev gamma function

$$\Gamma_\infty^-(\alpha) := \int_\mathbb{C} (N(z))^{\alpha-1} \exp(-2\pi i \text{Tr}(z)) |dz \wedge d\bar{z}|,$$

where N and Tr denote the Galois norm and trace of $\mathbb{C}$ over $\mathbb{R}$ and for which we also have $\Gamma_\infty^-(\alpha)\Gamma_\infty^-(1-\alpha) = 1$. We can express it as $\Gamma_\infty^-(\alpha) = (2\pi)^{1-2\alpha}\pi^{-1}\Gamma^2(\alpha)\sin\pi\alpha$. In a similar fashion, the Gel'fand-Graev beta function is defined on $\mathbb{C}$ and analogous symmetry properties which yield new functional equations can be derived ([10]).

B. Gamma functions and Fourier transforms on $\mathbb{Q}_p$ and $\mathbb{Q}_p(\sqrt{D})$

For the p-adic field $\mathbb{Q}_p$, we define a Fourier transform $\tilde{\phi}(\xi)$ of a test function $\phi(x)$, by

$$\tilde{\phi}(\xi) := \int_{\mathbb{Q}_p} \phi(x)\chi_p(x\xi) d\mu_p,$$

and the inverse Fourier transform by

$$\phi(x) := \int_{\mathbb{Q}_p} \tilde{\phi}(\xi)\chi_p(-x\xi) d\mu_p.$$

Here, $d\mu_p$ stands for the usual Haar measure on $\mathbb{Q}_p$ which it is granted to exist on locally compact groups, such as $\mathbb{Q}_p$, and that we will take with the normalization given by $\mu_p(\mathbb{Z}_p) = 1$. Moreover, $\chi_p(t) := \exp(2\pi i \{t\}_p)$ denotes the additive character of $\mathbb{Q}_p$. Now, the Gamma function we will use for the Fourier transform of $|x|_p^{\alpha-1}$ will be

$$\Gamma_p(\alpha) := \int_{\mathbb{Q}_p} |x|_p^{\alpha-1}\chi_p(x) d\mu_p = \frac{1-p^{\alpha-1}}{1-p^{-\alpha}}.$$

We will flesh out this calculation, to show how p-adic integrals work, and the rest of those will be generalizations of those same arguments. Some of them can be found in [1]. We begin by making use of the decomposition of $\mathbb{Q}_p$ we already cited to claim that

$$\int_{\mathbb{Q}_p} |x|_p^{\alpha-1}\chi_p(x)d\mu_p = \sum_{m=-\infty}^{\infty} p^{(1-\alpha)m}\int_{p^m\mathbb{Z}_p\setminus p^{m+1}\mathbb{Z}_p}\chi_p(x)d\mu_p.$$

This is due to the previous decomposition of $\mathbb{Q}_p$ and it helps to factor out the integral of the p-adic absolute value for each of those sets. Now we will see, using the fact that for those sets the character is nontrivial, that for $m\leq -2$ the integrals all vanish, because notice that in this case, $p^m\mathbb{Z}_p\setminus p^{m+1}\mathbb{Z}_p = \frac{1}{p}+(p^m\mathbb{Z}_p\setminus p^{m+1}\mathbb{Z}_p)$. Then, using a change of variable we put

$$\int_{p^m\mathbb{Z}_p\setminus p^{m+1}\mathbb{Z}_p}\chi_p(x)d\mu_p(x) = \exp\left(\frac{2\pi i}{p}\right)\int_{p^m\mathbb{Z}_p\setminus p^{m+1}\mathbb{Z}_p}\chi_p(y)d\mu_p(y),$$

whence we deduce that the integral is 0. For $m=-1$ the integral is equal to $-p^{\alpha-1}$, because it is zero on $\frac{1}{p}\mathbb{Z}_p$, but trivial on $\mathbb{Z}_p$ as they are compact groups themselves. For $m\geq 0$ the character is trivial and the measure of the set is simply $\frac{1}{p^m}-\frac{1}{p^{m+1}}$ again because of the group action invariance.
The sum we are left with is geometric and hence we have the desired result. We can check that $\Gamma_p(\alpha)\Gamma_p(1-\alpha)=1$ as well. Now proceeding similarly we can define the p-adic beta functions.
In a field $\mathbb{Q}_p(\sqrt{D})$ ($D\notin\mathbb{Q}_p^{*2}$), we will normalize the Haar measure with $\int_{|z\bar{z}|_p\leq 1} d\mu_p = 1$ except for the case $p=2, D=5 \pmod 8$ when it will be 2. This situation will be summarised with $\delta_{D,p}$. In such a field, we will use $z=x+y\sqrt{D}$ and $\bar{z}$ the Galois conjugation of z. The Fourier transform is given by

$$\tilde{\varphi}(\theta) = \sqrt{|4D|_p}\int_{\mathbb{Q}_p(\sqrt{D})}\varphi(z)\chi_p(\mathrm{Tr}(z\cdot\theta))d\mu_p(z),$$

where $\theta=\xi+\eta\sqrt{D}$. The inverse Fourier transform is similar but with the sign changed for the character χ_p. The Gamma function for this field is given by

$$\Gamma_p^{(D)}(\alpha) = \sqrt{|4D|_p}\int_{\mathbb{Q}_p(\sqrt{D})}|N(z)|_p^{\alpha-1}\chi_p(\mathrm{Tr}(z))d\mu_p(z).$$

The relation $\Gamma_p^{(D)}(\alpha)\Gamma_p^{(D)}(1-\alpha)=1$ also holds for this Gamma function. The construction of the beta function and Gel'fand-Graev beta function for $\mathbb{Q}_p(\sqrt{D})$ is analogous but with adequate normalization factors.

C. Adeles, ideles, their Haar measure and Fourier transform

The normalization we will choose for the adele group $\mathbb{A}_D$ will be the following:

$$\int_{\mathbb{A}_D^0} d\mu_D(z) = \begin{cases} 2, & \text{if } D\equiv 5 \pmod 8,\\ 1, & \text{if } D\not\equiv 5 \pmod 8,\end{cases}$$

where $\mathbb{A}_D^0$ is the compact set of adeles (with the previously introduced notation) with $0\leq x_\infty, x'_\infty\leq 1$ if $D>0$ and $z_\infty\bar{z}_\infty\leq\frac{1}{2\pi}$ if $D<0$, and $x_p, x'_p\in\mathbb{Z}_p$ for $p\in P_D^+$ and $z_p\in\mathbb{Z}_p(\sqrt{D})$ for $p\in P_D^-$. This measure can be seen formally as an infinite product

$$d\mu_D(z) = d\mu_{D,\infty}(z_\infty)\prod_{p=2}^{\infty}d\mu_{D,p}(z_p).$$

This allows us to interpret the integral as an infinte product of known integrals for each p if the integrable function is of the form of product for each prime and the infinity prime. Similar definitions with suitable normalizations arise for $\mathbb{A}_D^*$ which is also a locally compact group.
If we consider **standard functions** on $\mathbb{A}_D$ as finite linear combinations of products for each p of integrable distributions over the corresponding space, for which there exists a P such that for all $p>P$ it is $\varphi_p(x_p)=\varphi_p(x'_p)=\Omega(|x|_p)$ and $\varphi_p(z_p)=\Omega(|z_p\bar{z}_p|_p)$, where $\Omega(t)=\chi_{[0,1]}(t)$, the Fourier transform is defined as

$$\tilde{\varphi}(\zeta) := \frac{1}{2\sqrt{|D|}}\int_{\mathbb{A}_D^0}\varphi(z)\chi_D(z\zeta)d\mu_D(z),$$

where the character $\chi_D(z)$ is defined by the product of

$$\varepsilon(D)\prod_{p\in P_D^+}\chi_p(x_p)\chi_p(x'_p)\prod_{p\in P_D^-}\chi_p(z_p+\bar{z}_p),$$

where

$$\varepsilon(D) = \begin{cases}\chi_\infty(x_\infty)\chi_\infty(x'_\infty), & D>0,\\ \chi_\infty(z_\infty+\bar{z}_\infty), & D<0.\end{cases}$$

Notice the fact that $\prod_{p=2}^{\infty}\sqrt{|4D|_p} = \frac{1}{2\sqrt{|D|}}$.

D. Mellin transforms and Tate's formula

For Re $\alpha>0$, we define the **Mellin transform** of a complex integrable distribution φ defined on $\mathbb{R}$ as

$$\Phi_\infty(\alpha) = \int_{-\infty}^{\infty}\varphi(x)|x|^{\alpha-1}dx,$$

and we will consider its analytic continuation for $\alpha\notin -2\mathbb{N}$ and zero. We have $\Phi_\infty(\alpha)=\Gamma_\infty(\alpha)\tilde{\Phi}(1-\alpha)$ in those cases, cf. [1]. For a complex integrable distribution φ defined on $\mathbb{C}$ the Mellin transform is defined as

$$\Phi_\infty^-(\alpha) = \int_{\mathbb{C}}\varphi(z)(N(z))^{\alpha-1}|dz\wedge d\bar{z}|,$$

and we have $\Phi_\infty^-(\alpha)=\Gamma_\infty^-(\alpha)\tilde{\Phi}_\infty^-(1-\alpha)$ when $\alpha\notin -\mathbb{N}$. For a complex integrable distribution φ defined on $\mathbb{Q}_p$ the transform is defined as

$$\Phi_p(\alpha) = \frac{1}{1-p^{-1}}\int_{\mathbb{Q}_p}\varphi(x)|x|_p^{\alpha-1}d\mu_p(x).$$

And for a complex integrable distribution φ_D defined on $\mathbb{Q}_p(\sqrt{D})$ with $D \notin \mathbb{Q}_p^{*2}$, we define the transform by

$$\Phi_p^{(D)}(\alpha) = \frac{1}{\delta(1-q^{-1})} \int_{\mathbb{Q}_p(\sqrt{D})} \varphi_D(z)|N(z)|_p^{\alpha-1} d\mu_p(z).$$

The formulae $\Phi_p^{(D)}(\alpha) = \Gamma_p^{(D)}(\alpha)\tilde{\Phi}_p^{(D)}(1-\alpha)$ hold, where $\tilde{\Phi}_p^{(D)}$ stands for the Mellin transform of $\tilde{\varphi}$.
Now if φ is a standard function on $\mathbb{A}_D$, its Mellin transform is defined by

$$\Phi(\alpha) = \int_{\mathbb{A}_D} \varphi(z)|z|_D^{\alpha-1} d\mu_D(z).$$

This can be rewritten in the form of products using previous definitions; cf. [1]. Lastly we state the key theorem that will permit to obtain the needed regularizations.
Tate's theorem: *The function $\Phi(\alpha)$ can be analytically continued to all $\mathbb{C}$ except for the poles at $\alpha = 0, 1$ with residues $\varphi(0)$ and $\tilde{\varphi}(0)$ respectively. Moreover the following functional relation holds:*

$$\Phi(\alpha) = \tilde{\Phi}(1-\alpha), \quad \alpha \neq 0, 1.$$

IV. REGULARIZATION OF PRODUCT OF AMPLITUDES

To begin this section we will prove a couple of adelic formulae needed.

A. First adelic formula

We will deduce the following functional equation for the Riemann zeta function: for any $P = \infty, 2, 3, \ldots$

$$\prod_{p=2}^{P} \Gamma_p(\alpha) \mathrm{AC} \prod_{p>P} \frac{1}{1-p^{-\alpha}} = \frac{\zeta(\alpha)}{\zeta(1-\alpha)} \mathrm{AC} \prod_{p>P} \frac{1}{1-p^{\alpha-1}},$$

holds, where AC stands for analytic continuation with respect to α. To do so we will use Tate's theorem to ensure the existence of analytic continuation.
Proof: Notice that, if we take the equality at $0 < \mathrm{Re}\ \alpha < 1$ without analytic continuation, the formula evidently holds by the expression of the zeta function as an Euler product. On the other hand, we notice that since this is given by products of gamma functions, it can be seen as a Mellin transform. Consequently, we can claim by Tate's theorem that the formula holds adding AC on each side so long as $\alpha \neq 0, 1$, since it works well for finite products and they will be equal by the unicity of the holomorphic extension.

B. Second adelic formula

For this formula we will use the Dedekind zeta function $\zeta_D(\alpha)$ of the field $\mathbb{Q}(\sqrt{D})$, defined for Re $\alpha > 1$ as

$$\zeta_D(\alpha) := \prod_{p=2}^{\infty} (1-q^{-\alpha})^{-v}.$$

This can be rewritten in terms of products for P_D^+ and P_D^- where v is 2 if D is a square in $\mathbb{Q}_p^{*2}$ and 1 otherwise. The adelic formula we want is, for $P = 2, 3, \ldots,$

$$\prod_{p=2}^{P} \Gamma_q^v(\alpha) \mathrm{AC} \prod_{p>P} (1-q^{-\alpha})^{-v} =$$

$$= \frac{\zeta_D(\alpha)}{\zeta_D(1-\alpha)} \mathrm{AC} \prod_{p>P} (1-q^{\alpha-1})^{-v}.$$

To prove this formula, it is simply a matter of realizing that we just need to extend the previous computation, that is, notice that again, for $0 < \mathrm{Re}\ \alpha < 1$, the products without analytic continuation make the formula holds where we have to take into account the sign of D. Next, we also make use of Tate's theorem to notice that we can see it as a Mellin transform, and therefore claim the existence of analytic continuation. The argument concludes in a similar fashion as the first adelic formula.
Denote by Δ_D the discriminant of the field $\mathbb{Q}(\sqrt{D})$. Now using the relations for the Dedekind zeta function, and the equality $\prod_{P_D^-} \rho_{D,p}(\alpha) = |\Delta_D|^{\alpha-\frac{1}{2}}$, we arrive at new representations of Γ_∞ formulae: $\Gamma_\infty^2(\alpha) = |\Delta_D|^{\frac{1}{2}-\alpha} \frac{\zeta_D(1-\alpha)}{\zeta_D(\alpha)}$, if $D > 0$, and when $D < 0$, $\Gamma_\infty^-(\alpha) = |\Delta_D|^{\frac{1}{2}-\alpha} \frac{\zeta_D(1-\alpha)}{\zeta_D(\alpha)}$. We then obtain adelic formulae for gamma functions of $\mathbb{Q}(\sqrt{D})$:

$$|\Delta_D|^{\frac{1}{2}-\alpha} \mathrm{AC} \prod_{p>P} (1-q^{\alpha-1})^{-v} =$$

$$= \prod_{p=2}^{P} \Gamma_q^v(\alpha) \mathrm{AC} \prod_{p>P} (1-q^{-\alpha})^{-v} \begin{cases} \Gamma_\infty^2(\alpha), & D > 0, \\ \Gamma_\infty^-(\alpha), & D < 0. \end{cases} \quad (1)$$

Similarly to the first adelic formula, those can be extended for beta functions of $\mathbb{Q}(\sqrt{D})$.

C. Regularization of divergent products

We will only do the case of gamma functions, since the beta functions case can be obtained slightly modifying the argument we will use for gamma functions. In the previous expression (1) we can make P go to infinity and obtain $|\Delta_D|^{\frac{1}{2}-\alpha} =$

$$= \lim_{P\to\infty} \prod_{p=2}^{P} \Gamma_q^v(\alpha) \mathrm{AC} \prod_{p>P} (1-q^{-\alpha})^{-v} \begin{cases} \Gamma_\infty^2(\alpha), & D > 0, \\ \Gamma_\infty^-(\alpha), & D < 0. \end{cases}$$

This allows us to define the regularized product of reduced gamma functions as

$$\mathrm{reg} \prod_{p=2}^{\infty} \Gamma_q^v(\alpha) = \lim_{P\to\infty} \prod_{p=2}^{P} \Gamma_q^v(\alpha) \mathrm{AC} \prod_{p>P} (1-q^{-\alpha})^{-v},$$

and its analytic continuation for the α for which the limit does not exist, where we note that the limit exists for Re $\alpha < 0$. Each product separately diverges, but not mulitplied together.

Now we can deduce that

$$|\Delta_D|^{\frac{1}{2}-\alpha} = \text{reg} \prod_{p=2}^{\infty} \Gamma_q^v(\alpha) \begin{cases} \Gamma_\infty^2(\alpha), & D > 0, \\ \Gamma_\infty^-(\alpha), & D < 0. \end{cases}$$

for $\alpha \neq 0, 1$. The case $D = 1$, then, becomes

$$\Gamma_\infty \text{reg} \prod_{p=2}^{\infty} \Gamma_p(\alpha) = 1.$$

We are thus in position to make sense of the divergent product proposed by Witten,

$$A(s,t,u) \prod_{p=2}^{\infty} A_p(s,t,u) = 1.$$

Here, $A(s,t,u)$ stands for an amplitude which is the object we compute in a scattering experiment and can be approximated, for closed or open strings, by the Veneziano and the Virasoro-Shapiro amplitudes, respectively. For the Veneziano approximation the amplitude and its p-adic analogues are given by

$$\begin{aligned} A(s,t,u) &:= B_\infty(-\alpha(s), -\alpha(t)), \\ A_p(s,t,u) &:= B_p(-\alpha(s), -\alpha(t)), \end{aligned} \quad p = 2, 3, \ldots,$$

where $B_p(\alpha, \beta) = \Gamma_p(\alpha)\Gamma_p(\beta)\Gamma_p(\gamma)$, and $\alpha + \beta + \gamma = 1$ for $p = \infty, 2, \ldots$. Furthermore, s, t, u are the previously mentioned Mandelstam variables, $s + t + u = -8$, for which $\alpha(s) = 1 + \frac{s}{2}$, and $-\alpha(s) - \alpha(t) - \alpha(u) = 1$. For the Virasoro-Shapiro case, we have amplitudes defined by $A^-(s,t,u) = B_\infty^-(-\alpha(s), -\alpha(t))$ and $A_p(s,t,u) = B_q(-\alpha(s), -\alpha(t))$, where the beta functions are given by $B_\infty^-(\alpha, \beta) = \Gamma_\infty^-(\alpha)\Gamma_\infty^-(\beta)\Gamma_\infty^-(\gamma)$ and $B_q(\alpha, \beta) = \Gamma_q(\alpha)\Gamma_q(\beta)\Gamma_q(\gamma)$, for which $s + t + u = -32$ and $\alpha(s) = 1 + \frac{s}{8}$.

In general, the regularization proposed that can be deduced from all the previous build-up is:

$$\left.\begin{aligned} D > 0, \ & A^2(s,t,u) \\ D < 0, \ & A^-(s,t,u) \end{aligned}\right\} \text{reg} \prod_{p=2}^{\infty} A_p^v(s,t,u) = \sqrt{|\Delta_D|}.$$

The case that motivated us is given by $D = 1$, where we obtain the Veneziano amplitude as

$$A(s,t,u) \ \text{reg} \prod_{p=2}^{\infty} A_p(s,t,u) = 1.$$

V. CONCLUSIONS

As seen in [8], p-adic string theory allows us to tackle problems of tachyon condensation and is a good toy model for the kind of results that we should obtain in string theory.

- We have seen an application in theoretical physics of a mathematical tool that was developed in Number Theory.
- We have been able to obtain a regularization for the adelic formula proposed by Witten and Freund so that their product formula makes sense.
- We have realized the importance of product formulae, which permits us to gain information of the euclidean case via the p-adic ones, that are easier to compute in the problem considered.

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